L07b Complexity of Computing NE

CS 280 Algorithmic Game Theory
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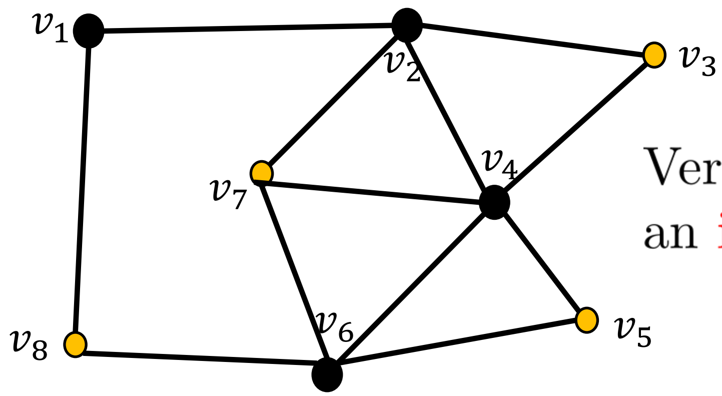
Inspired and some figures
by C. Daskalakis slides and T. Roughgarden notes

Warm-up: Reductions in NP

Example: **INDEPENDENT SET (IS) Problem**

Given a simple undirected graph $G(V, E)$ and k, is there an independent set in G of size $\geq\ k$. Independent set is called a set $I \subset V$ of vertices such that pairwise the vertices in I are not connected.



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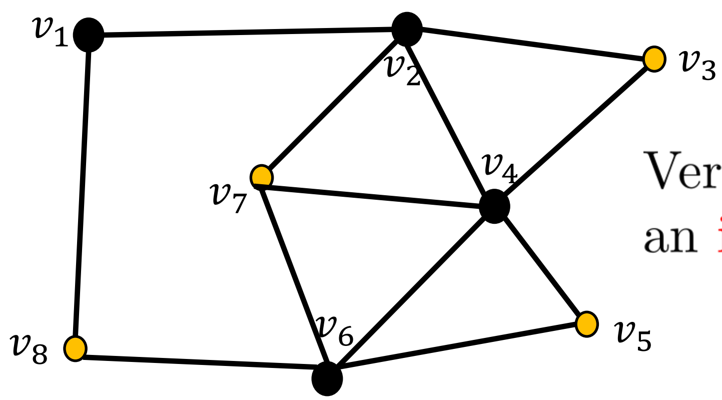

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Claim: INDEPENDENT SET is **NP-complete**.

Proof: (1) INDEPENDENT SET **belongs** to NP (why?).
(2) Reduce 3-SAT to INDEPENDENT SET. Since 3-SAT is NP-hard, INDEPENDENT SET is NP-hard.

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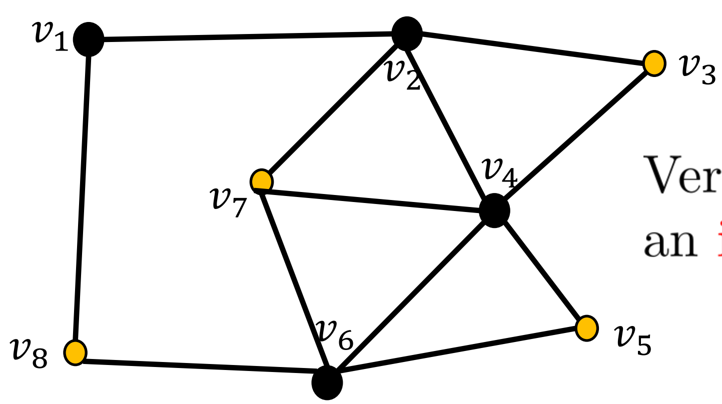

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(1), (2) imply IND. SET is NP-complete!

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A **literal** is a Boolean expression consisting of just a single Boolean variable, or the negation of a Boolean variable.

- Example: “$\neg x_1$” and “x_2” are literals.

A **clause** is a Boolean expression of the form “$\ell_1 \vee \ell_2 \vee \cdots \vee \ell_k$”, i.e. a **disjunction** of some literals $\ell_1, \ell_2, \dots, \ell_k$. In 3-SAT $k = 3$.

- Example: “$C_1 \equiv x_1 \vee \neg x_2 \vee x_3$” is a clause.

A Boolean expression is a conjunction of clauses.

Example: “$E \equiv C_1 \wedge C_2 \wedge C_3$” is a clause.

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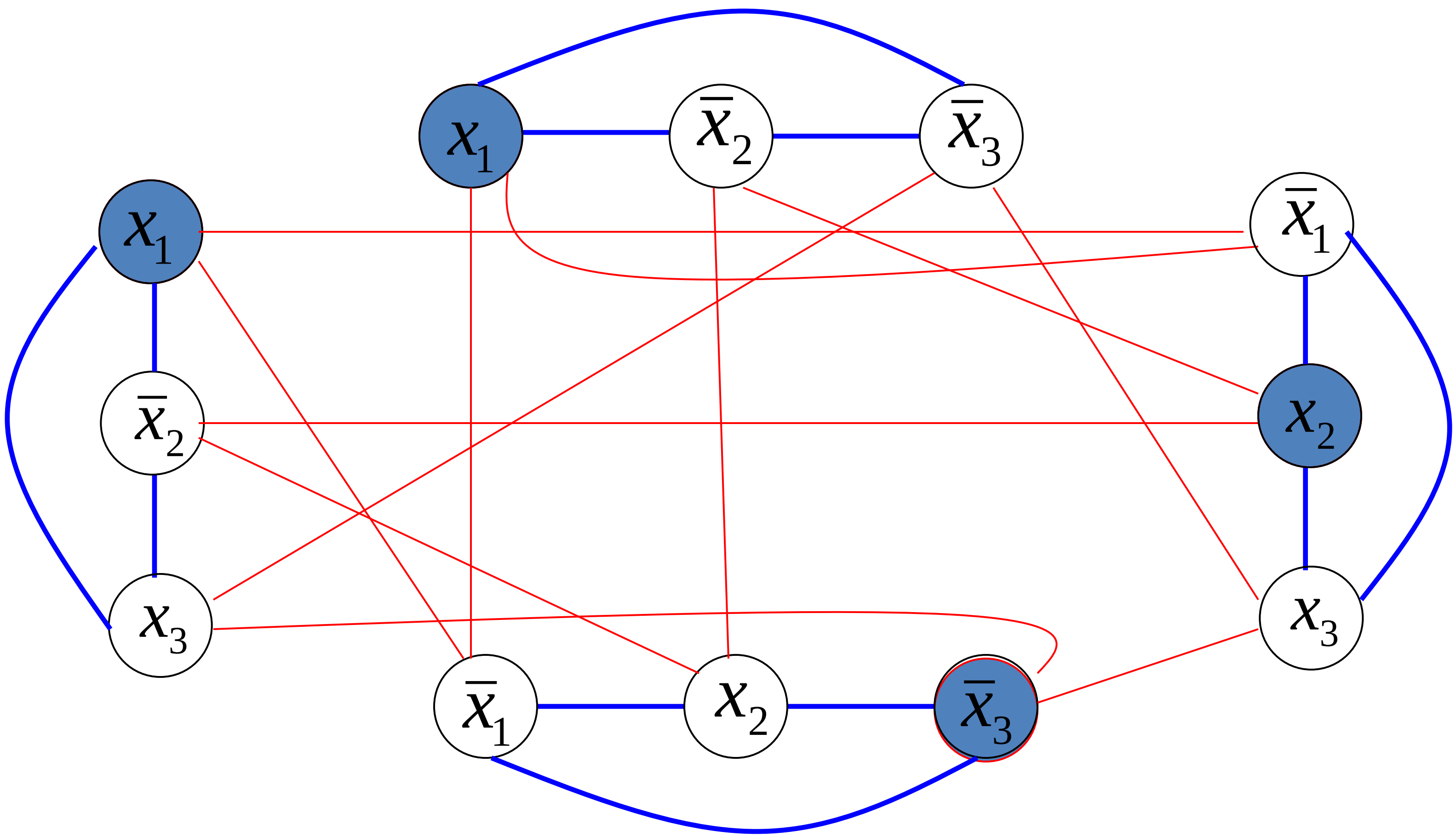

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Claim: Expression E with k clauses is satisfiable if and only if the induced graph G has an IS of size k.

Therefore, given a **graph G and a k**, if we can identify in **poly-time** if there exists an **Independent Set of size at least k**, then we can solve **in poly-time 3-SAT**.

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Question: Can the problem of computing a Nash Equilibrium be NP-complete?

Answer: (Megiddo) Suppose we have a reduction from SAT to NASH, s.t any solution to the instance of NASH tells us whether or not the SAT instance has a solution. Then we could turn this into a nondeterministic algorithm for verifying that an instance of SAT has no solution: Just guess a solution of the NASH instance, and check that it indeed implies that the SAT instance has no solution. NP = co-NP (unlikely).

The class PLS

PLS (Polynomial-time Local Search) is a complexity class intended to exemplify local search problems. An abstract local search problem is specified by three polynomial-time algorithms.

Canonical Problem: LOCAL MAX-CUT

Given an undirected graph $G = (V, E)$ with non-negative weights w_e on edges, find a cut $(S, \overline{S})$ that maximizes the total weight of cut edges. You are allowed to do only local moves that improve the objective, i.e., moving one vertex v from one side of the cut to the other that improves the total weight of cut edges.

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Remark: (classic) MAX-CUT is NP-Complete.

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3. The third algorithm takes as input an instance and a feasible solution and either reports "locally optimal" or produces a better solution (for LOCAL MAX-CUT it checks all possible $|V|$ moves. If one improves the objective choose that move).

Theorem (Local Max-cut is PLS-complete). *The LOCAL MAX-CUT problem is PLS-complete.*

The complexity of Pure Nash Eq.

Theorem (PNE in congestion games is PLS-complete)**.** *The problem of computing Pure Nash Equilibria in Congestion Games is PLS-complete.*

Proof. We show first that PNE CONGESTION GAMES $\in$ PLS.

Describe the three algorithms:

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- Second algorithm takes a congestion game and a strategy profile s, and returns the value of the potential function $\Phi(s) = \sum_e \sum_{j=1}^{l_e(s)} c_e(j)$.
- The third algorithm checks if the given strategy profile s is a PNE; if not, we find an agent i that deviates from s_i to another pure s_i' and decreases her utility. Then $\Phi(s_i', s_{-i}) < \Phi(s_i, s_{-i})$. This can be done polynomial time in the description of the game.

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- The cost $c_{r_e}/c_{\overline{r}_e}$ of a resource r_e or $\overline{r}_e$ is 0 if one agent uses it and w_e if two players use it.

This transformation is poly-time.

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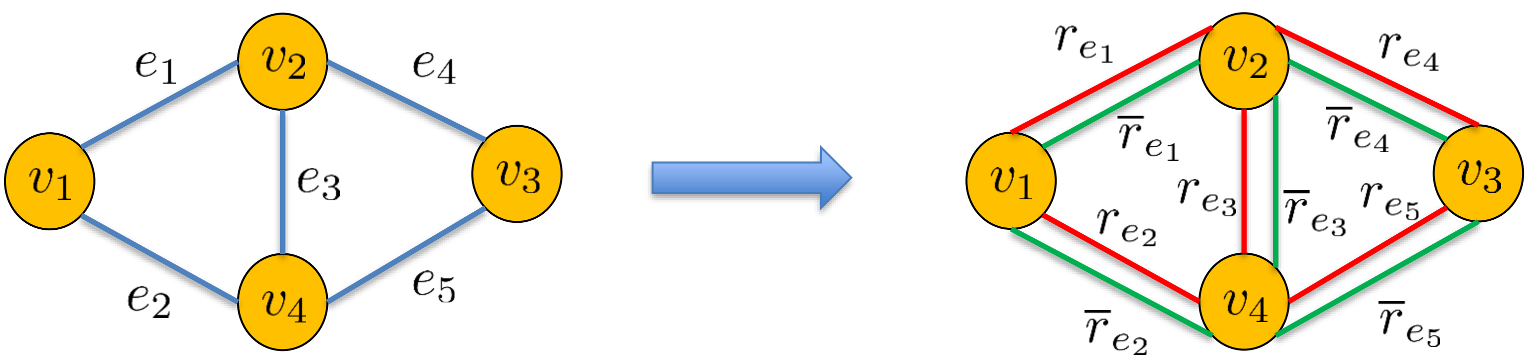

Each agent has two strategies, red and green.

Say agents v_1, v_2 choose red and v_3, v_4 choose green. Cost of v_1, v_2 is w_{e_1} and of v_3, v_4 is w_{e_5}.

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Therefore:

- Cuts with larger weight correspond to strategy profiles with smaller potential.
- Local maxima of cuts of G correspond to local minima of the potential function.

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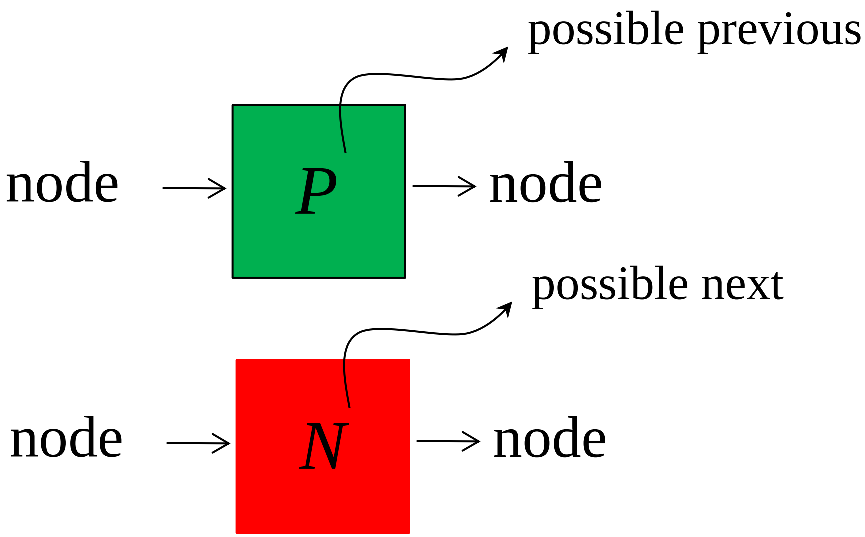
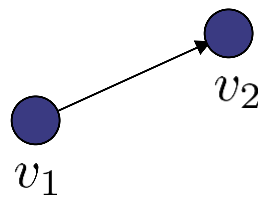
Suppose that an exponentially large graph with vertex set $\{0,1\}^n$ (i.e, 2^n vertices) is defined by two circuits:



Example:



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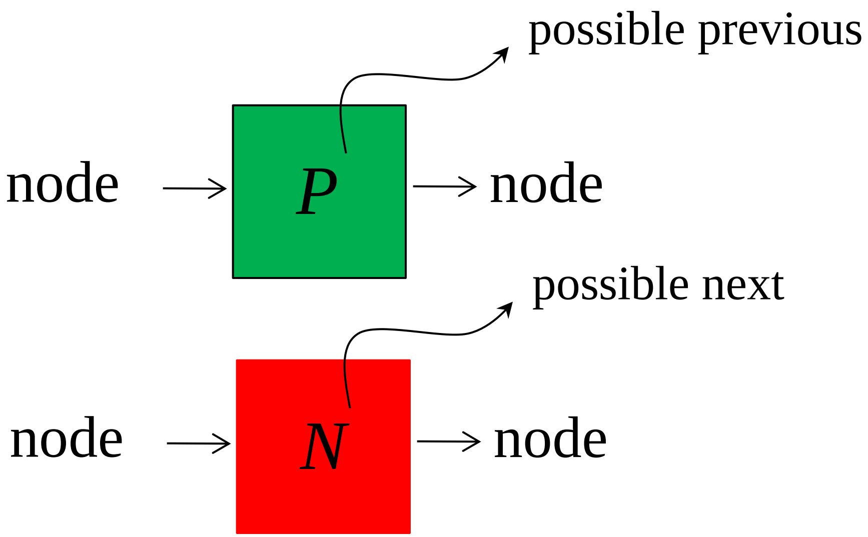

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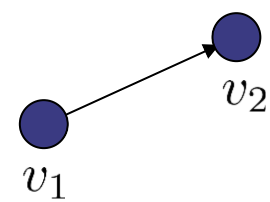

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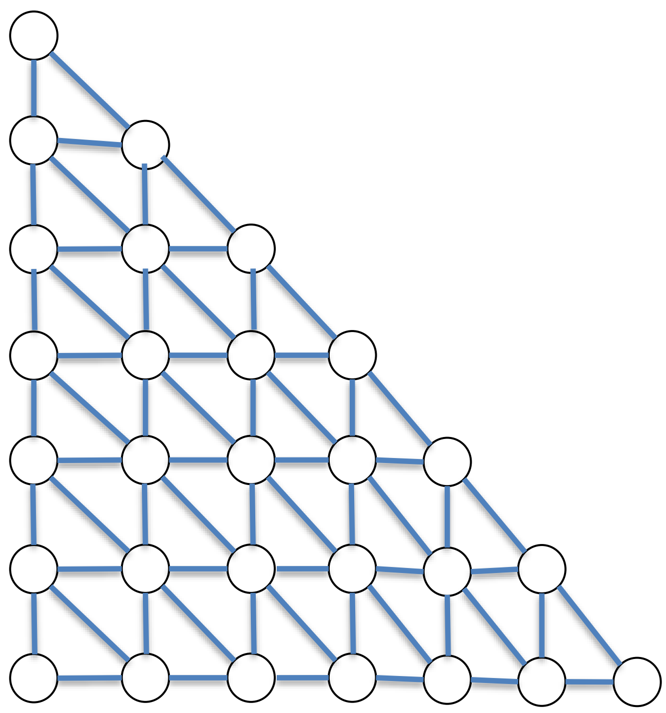
Canonical Problem:

END OF THE LINE: Given P, N: If 0^n is an unbalanced node, find another unbalanced node. Otherwise return 0^n.

PPAD (Papadimitriou 94'): All problems in FNP reducible to END OF THE LINE.

2D Sperner's Lemma

Theorem (A trichromatic triangle always exist). *Consider triangulation of 2d simplex Δ and a proper 3-coloring, that assign each vertex a different color and inside vertices on each edge of Δ use only the two colors of the respective endpoints. Then there always exists a trichromatic triangle (odd in number!).*



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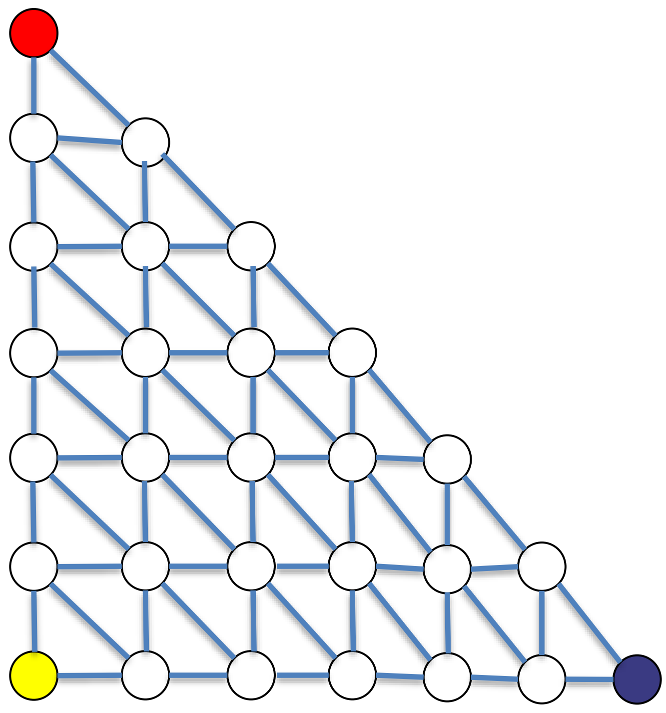

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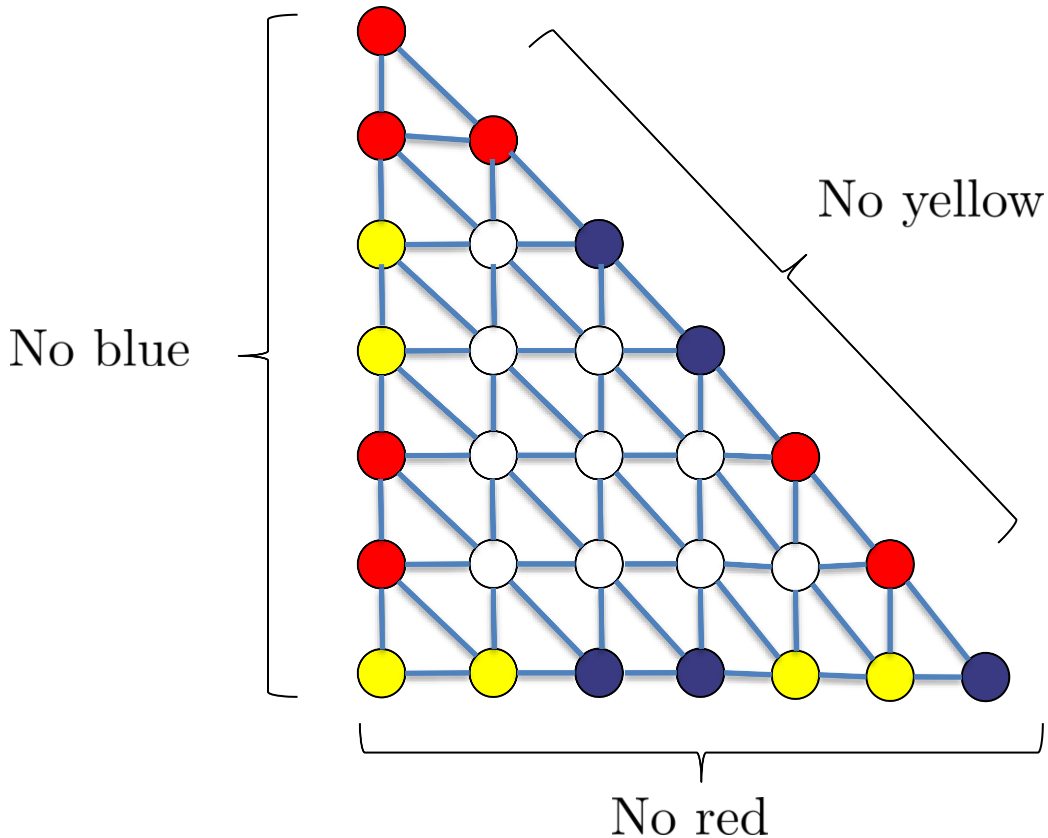

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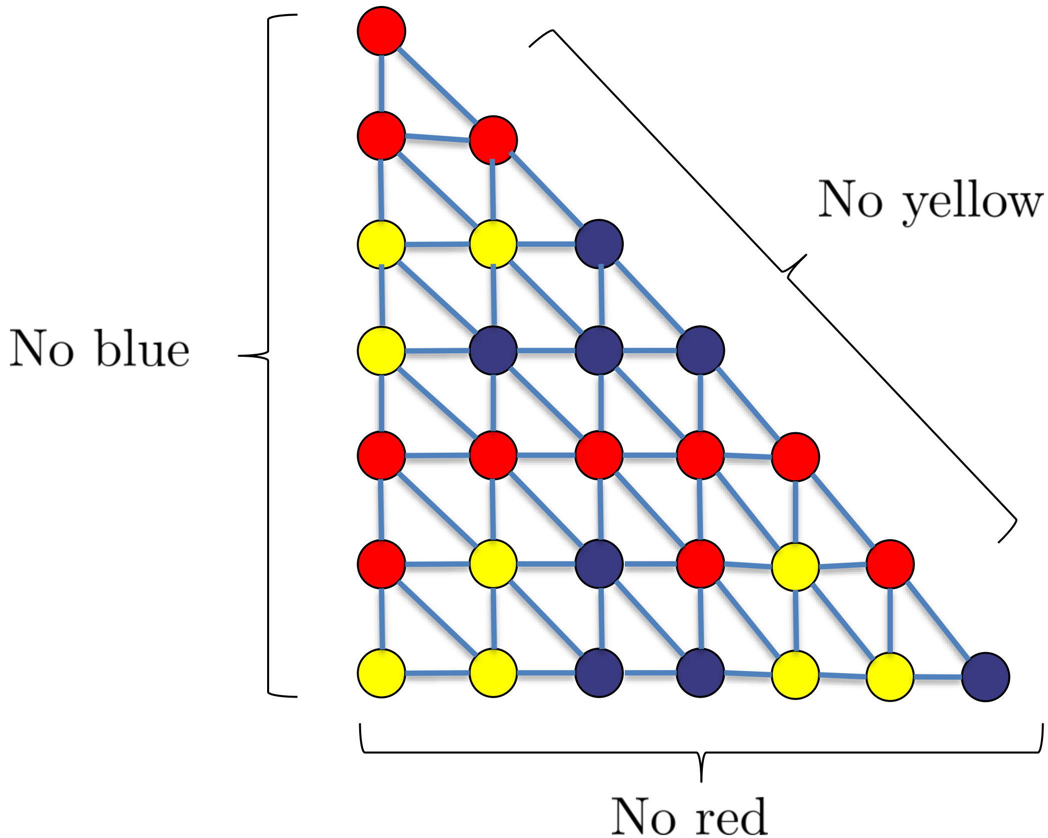

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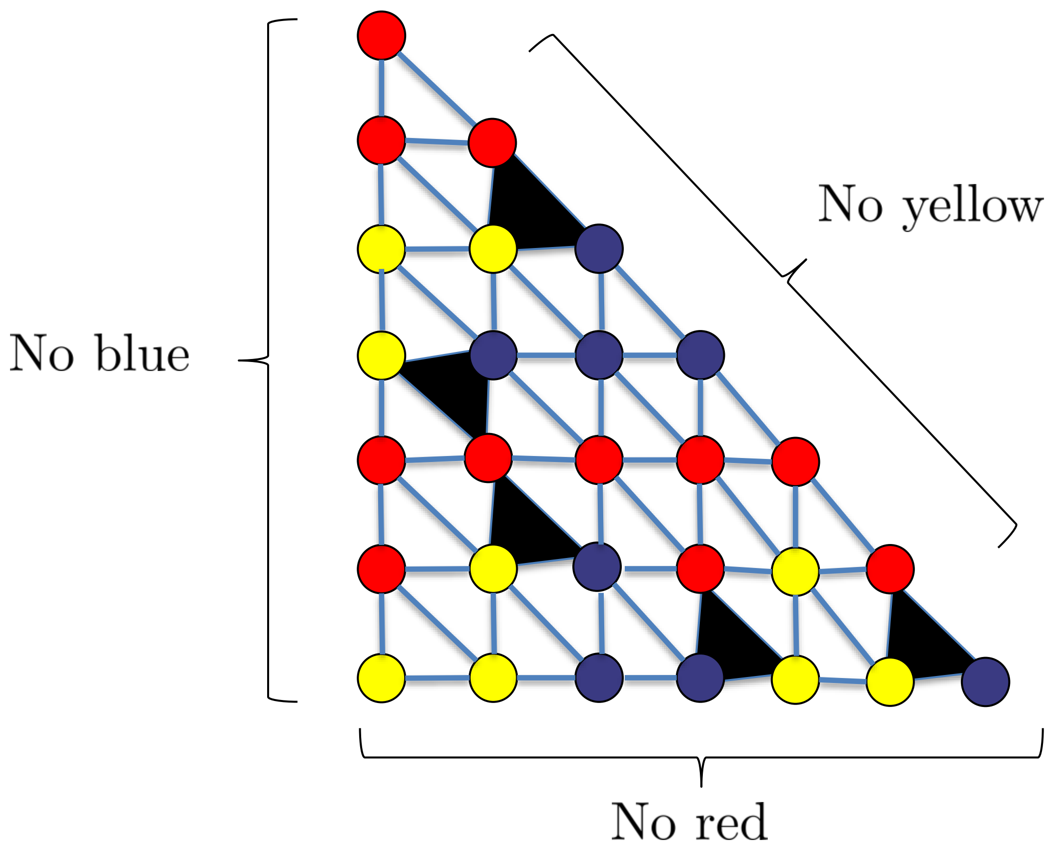

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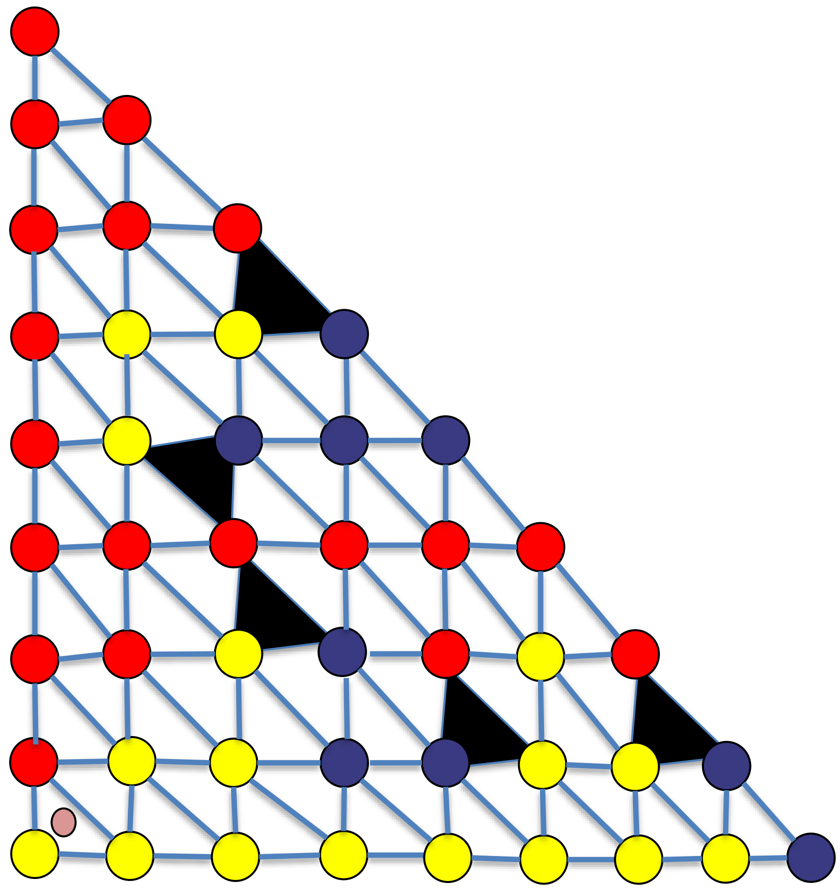
Proof. We introduce an outer boundary for conveniece that does not create new trichromatic triangles. Next we define a directed walk starting from the bottom-left triangle.



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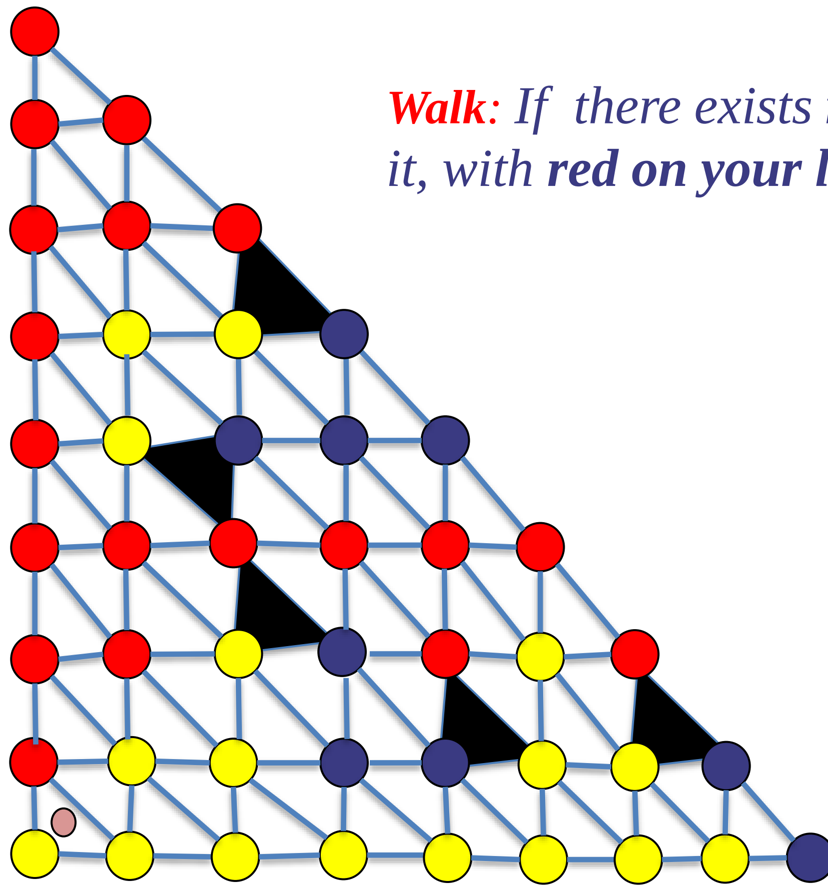

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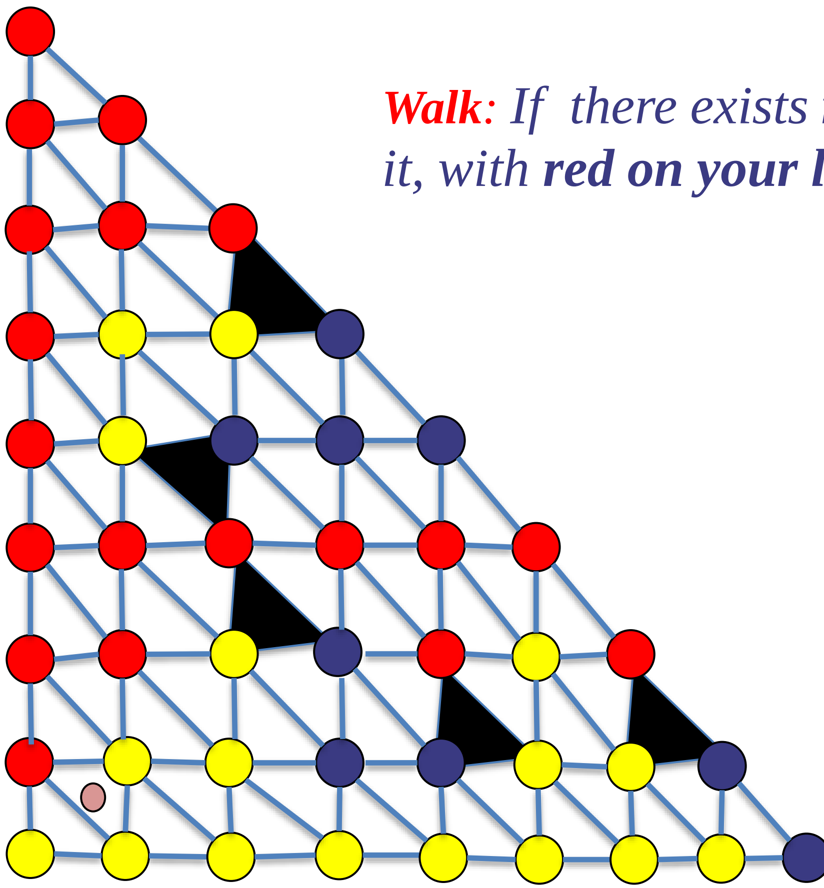

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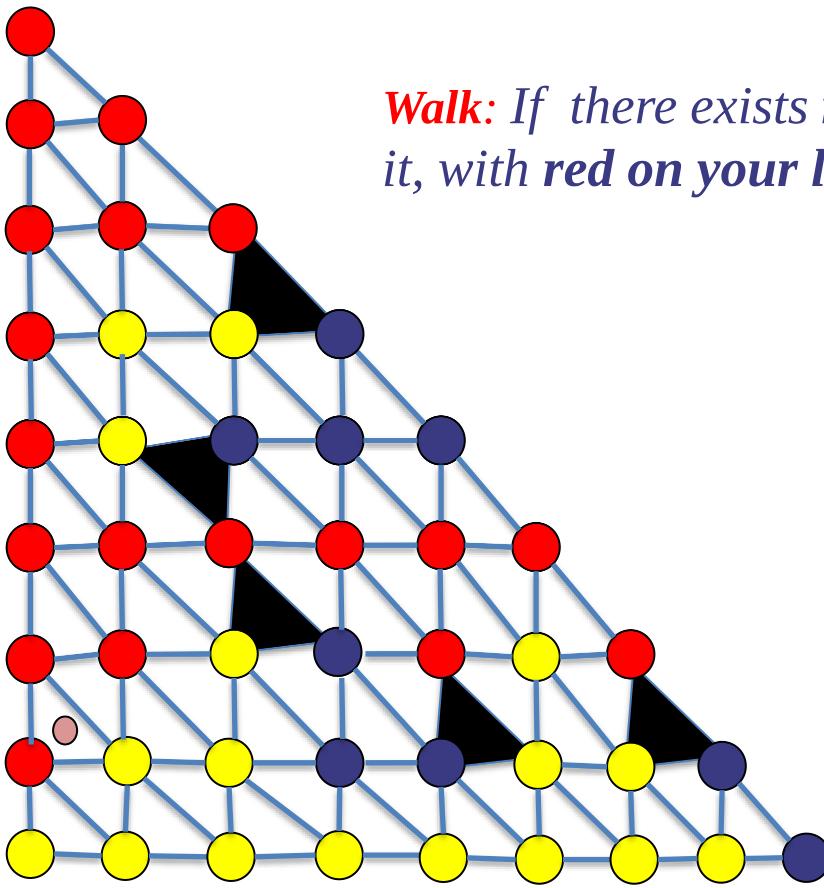

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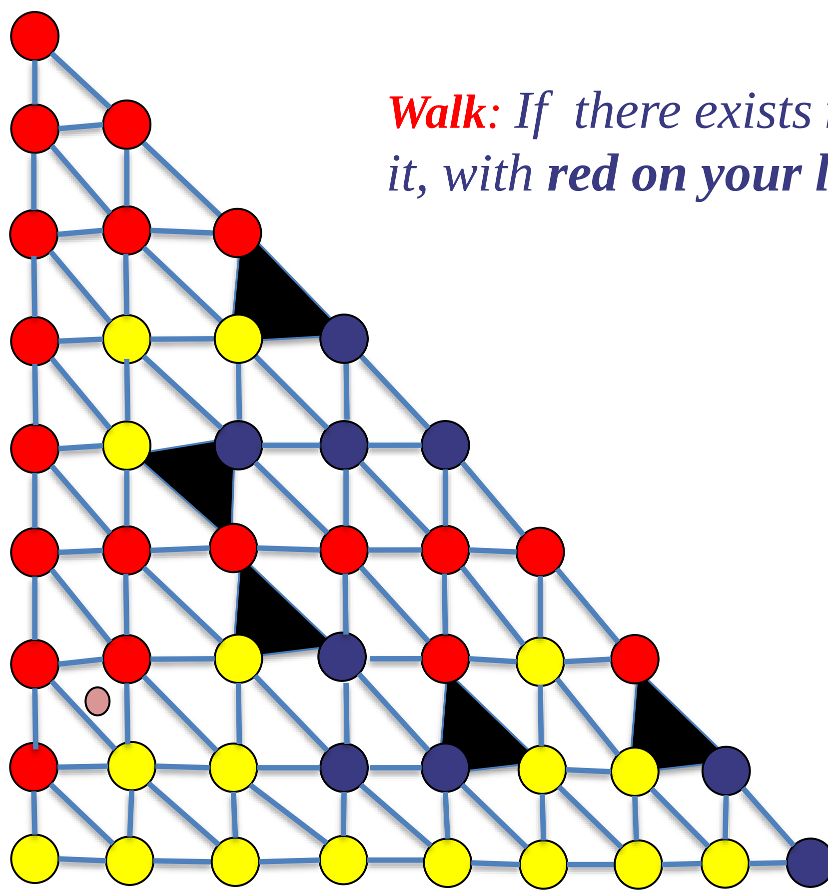

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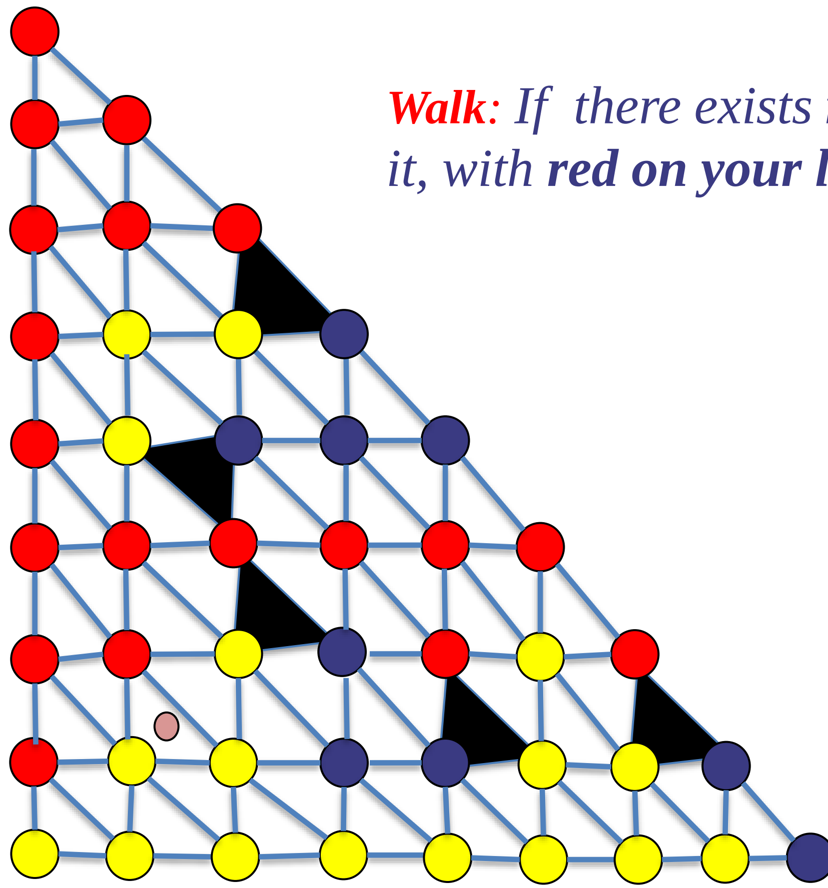

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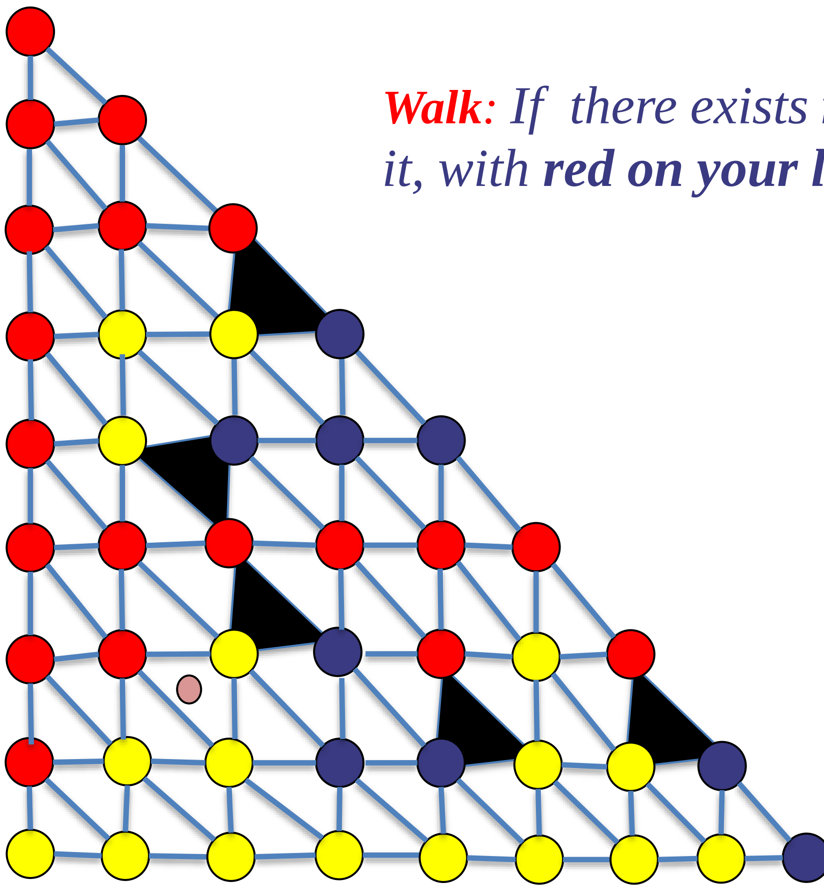

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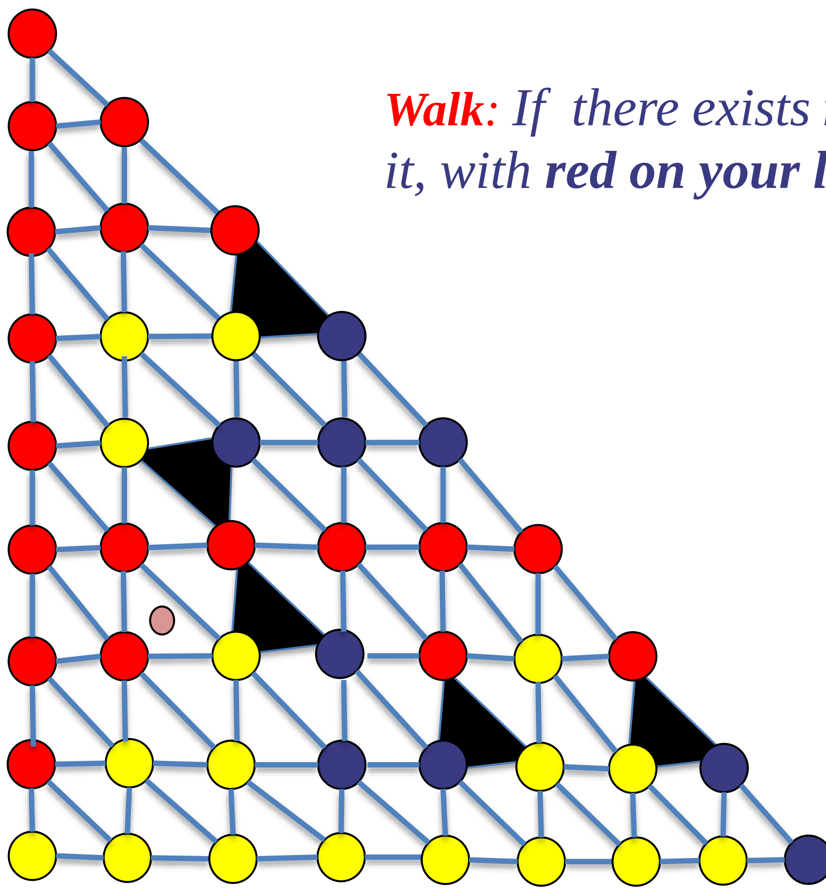

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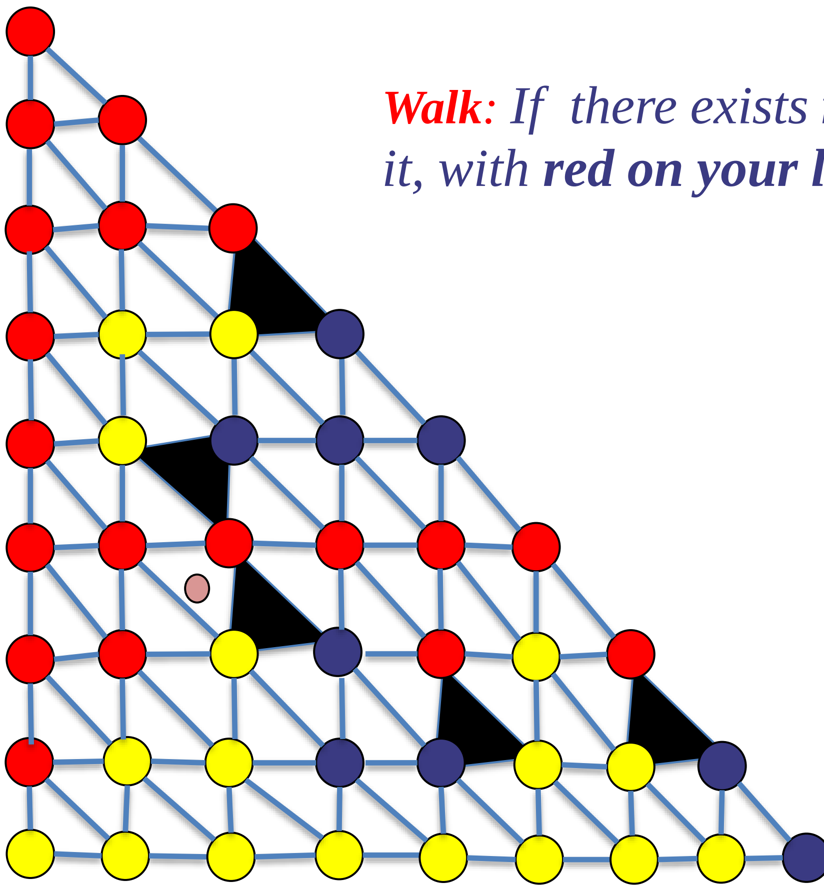

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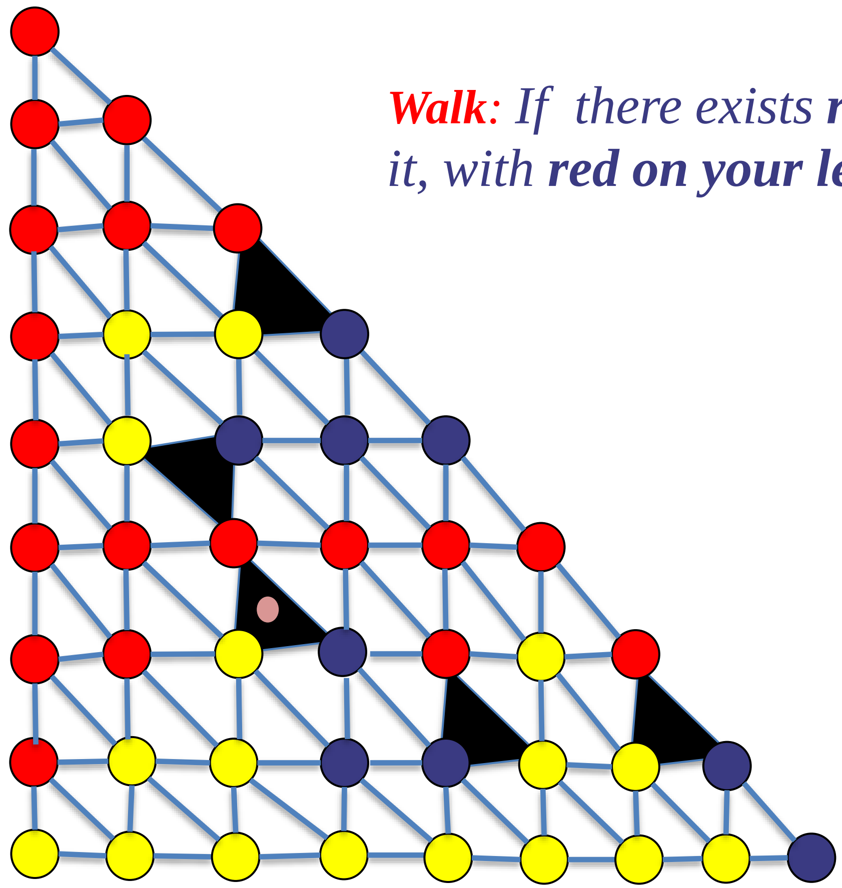

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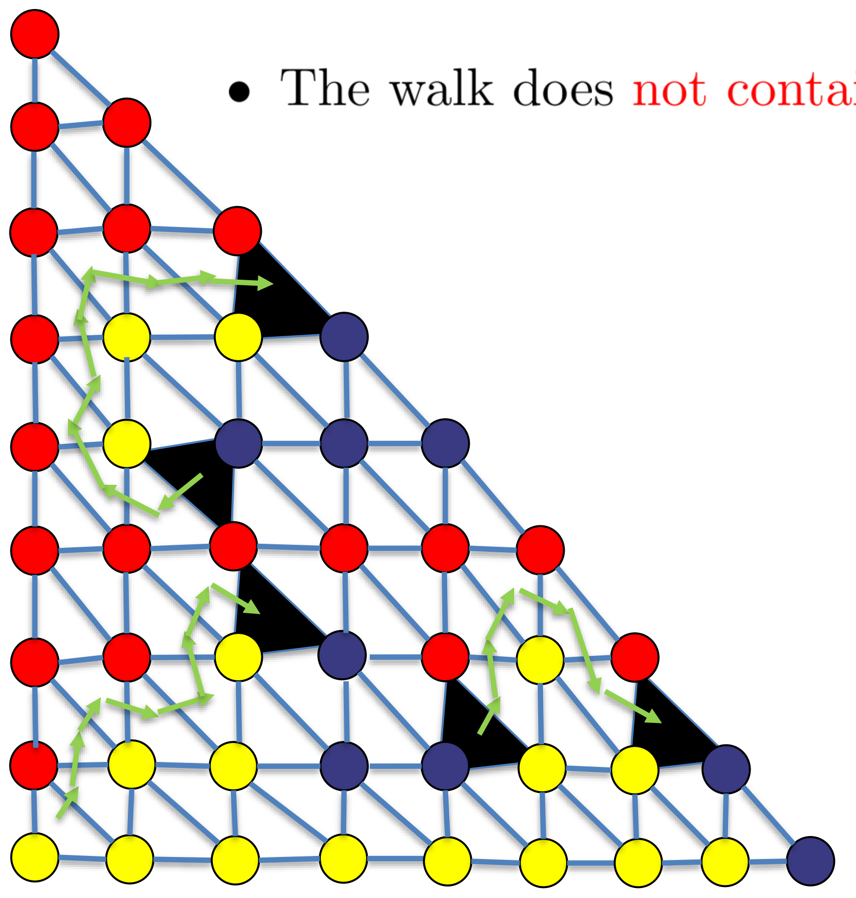

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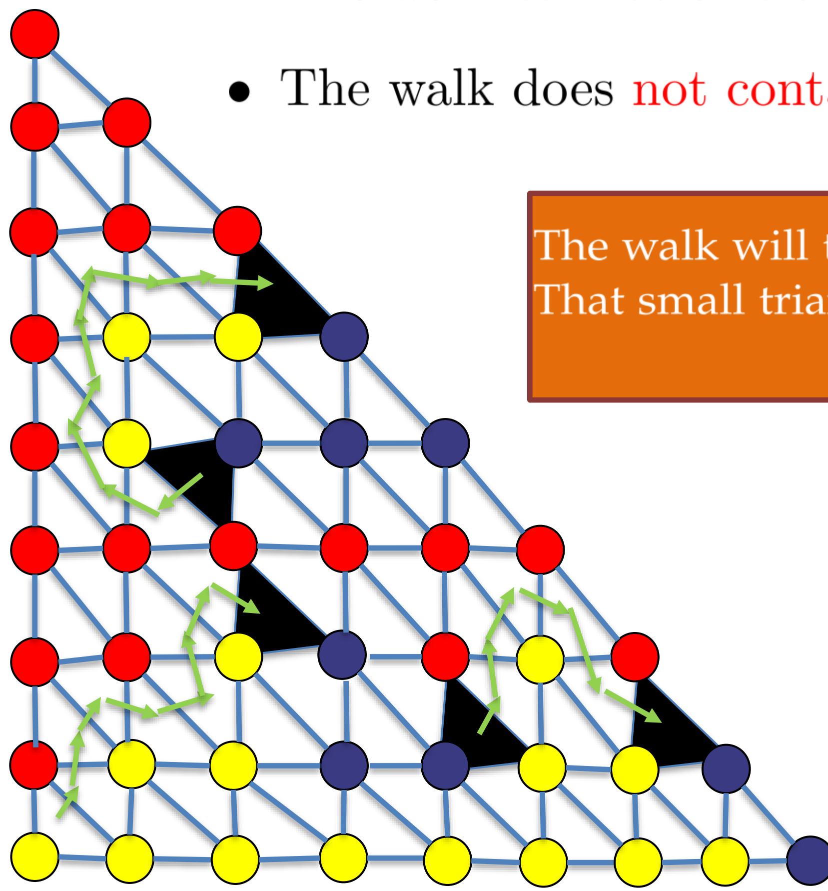

The walk will terminate incide somewhere!
That small triangle should be trichromatic!

2D Sperner's Lemma

Proof cont.

- The walk cannot exit the outer triangle (why?).
- The walk does not contain ρ shapes (why?).

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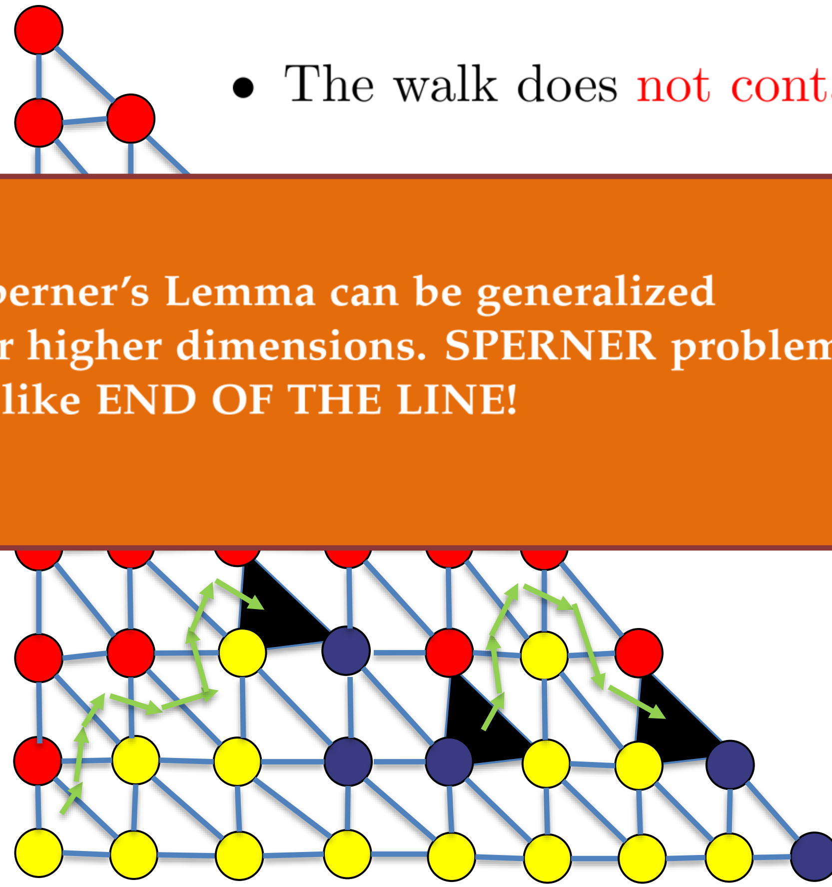

Sperner's Lemma can be generalized for higher dimensions. SPERNER problem is like END OF THE LINE!

BROUWER

Definition (BROUWER). *The problem BROUWER is defined below:*

Input: *A poly-time algorithm Π_F for the evaluation of a function $F : [0,1]^m \to [0,1]^m$, a constant K such that F is K-Lipschitz and accuracy ϵ.*

Output: A (rational) point x so that

$$\|F(x) - x\|_\infty \leq \epsilon,$$

i.e., x is an approximate fixed point.

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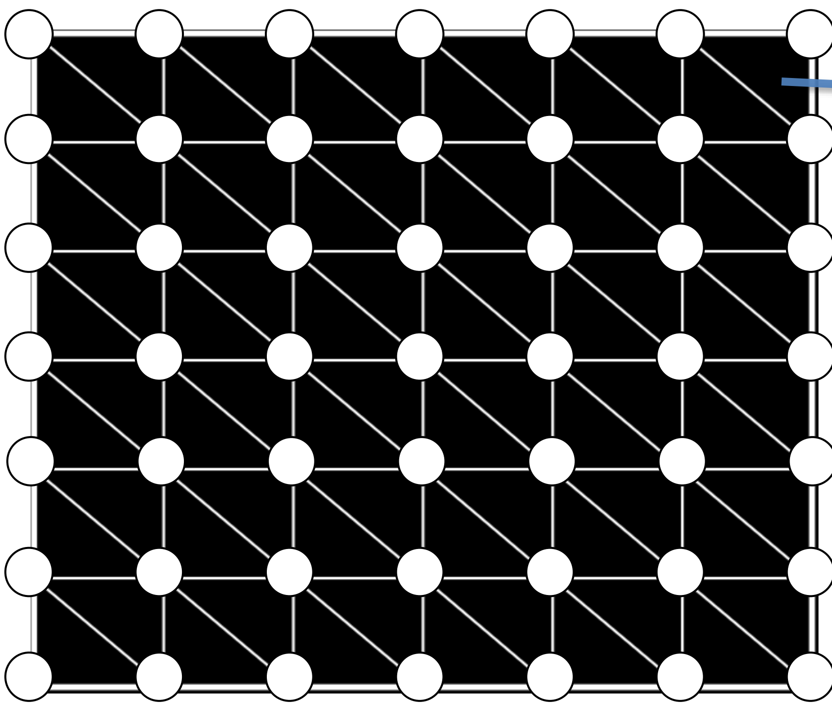
We will show that

$$\boxed{\text{BROUWER} \to \text{SPERNER}}$$

2D BROUWER reduction to SPERNER

Let $F : [0,1]^2 \to [0,1]^2$. By uniform continuity there exists a $\delta(\epsilon)$ so that

$$\|x - y\|_\infty \leq \delta \Rightarrow \|F(x) - F(y)\|_\infty \leq \epsilon.$$



Diameter of each cell is at most $\delta(\epsilon)$

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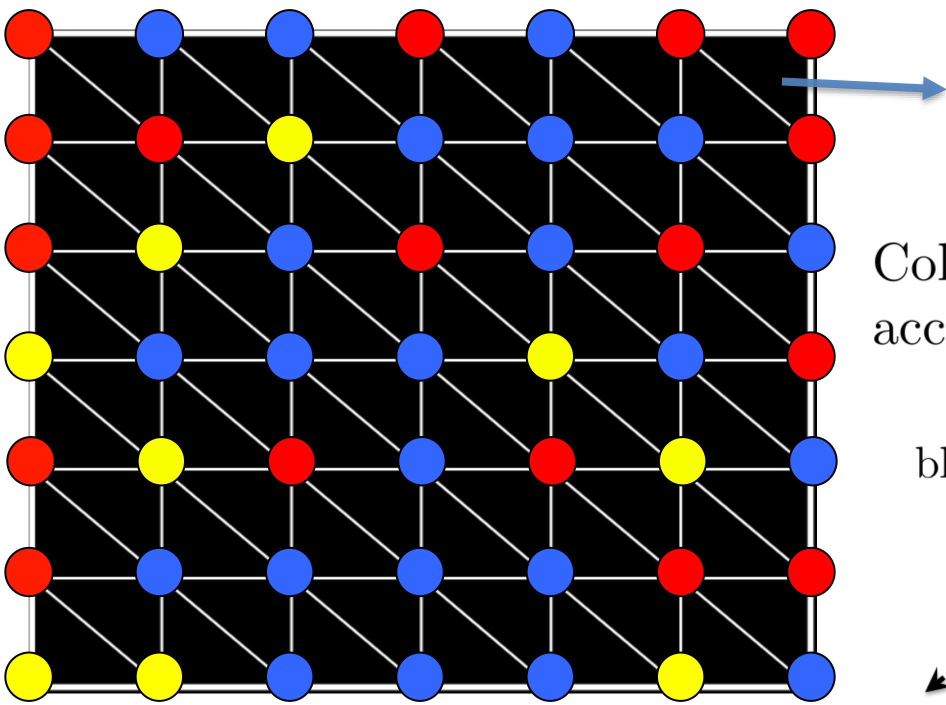

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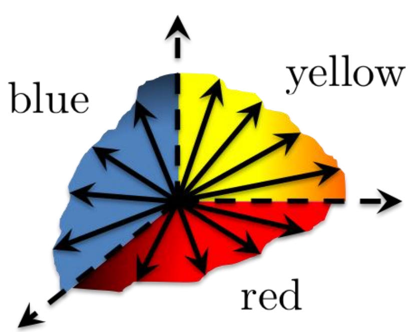
Color the nodes of the triangulation according to the direction of $f(x) - x$.



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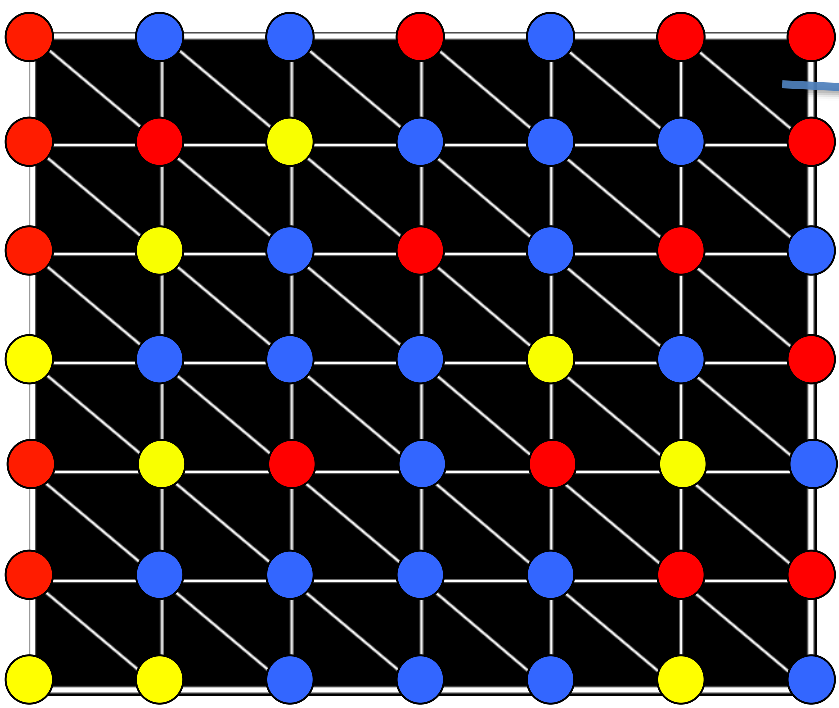

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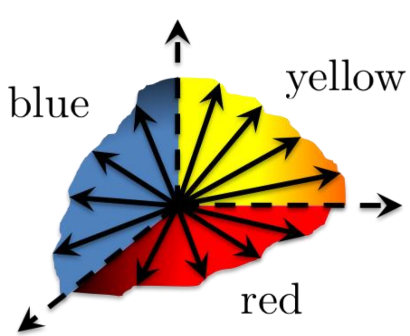
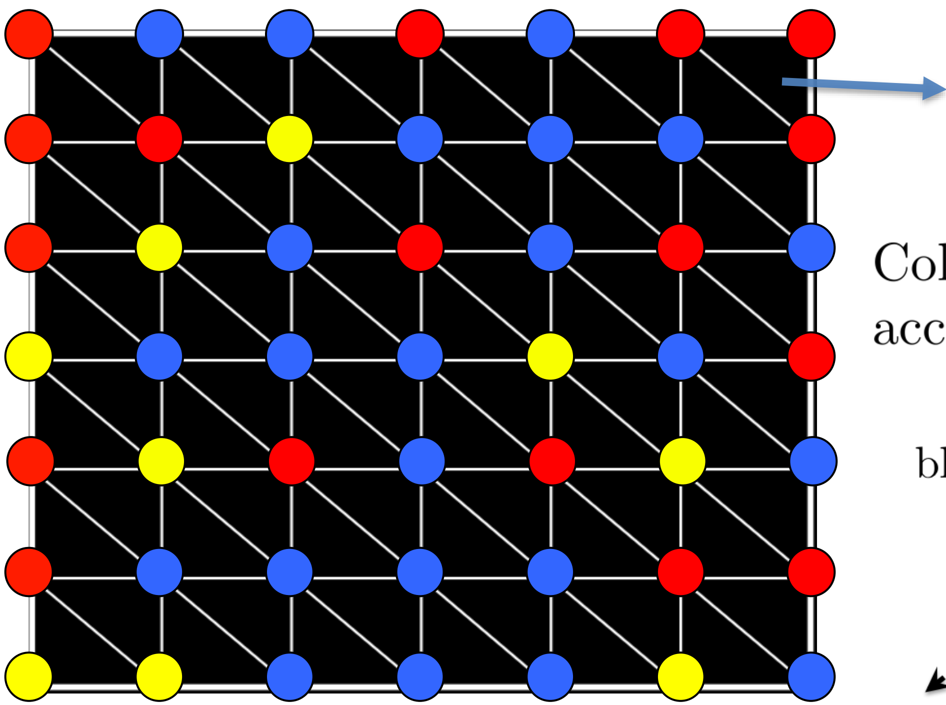


Tie-break at the boundary angles, so that the resulting coloring respects the boundary conditions!

2D BROUWER reduction to SPERNER

Claim. *Choose* $\delta = \min(\delta(\epsilon), \epsilon)$ *and let* v^y *be the yellow vertex of a trichromatic triangle. It holds that*

$$\|F(v^y) - v^y\|_\infty \leq 2\epsilon.$$



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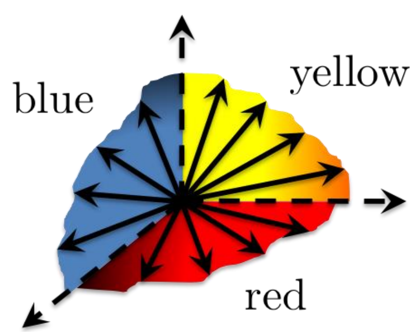

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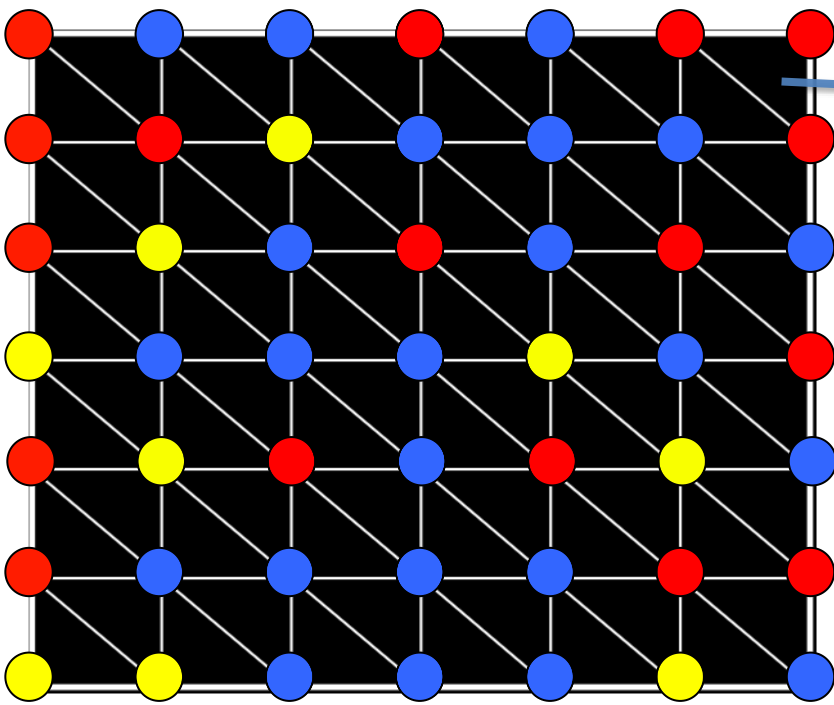
2D BROUWER reduction to SPERNER

Claim. *Choo* … *ellow*
vertex of a tri …

This will be in HW2.

$\|f(v^*) - v^*\|_\infty \le 2\epsilon.$



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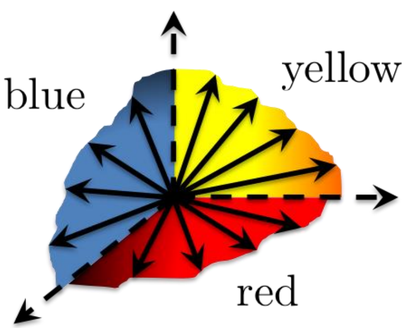

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NASH reduction to BROUWER

We will not see the proof, just an idea.

NASH reduction to BROUWER

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Consider the 2×2 mathcing pennies.

	H	T
H	1, -1	-1,1
T	-1, 1	1,-1

Consider the function f from the proof of Nash.

$$\boxed{f_{is_i}(x) = \frac{x_i(s_i) + \max\{u_i(s_i; x_{-i}) - u_i(x), 0\}}{1 + \sum_{s' \in S_i} \max\{u_i(s'; x_{-i}) - u_i(x), 0\}}}$$

NASH reduction to BROUWER

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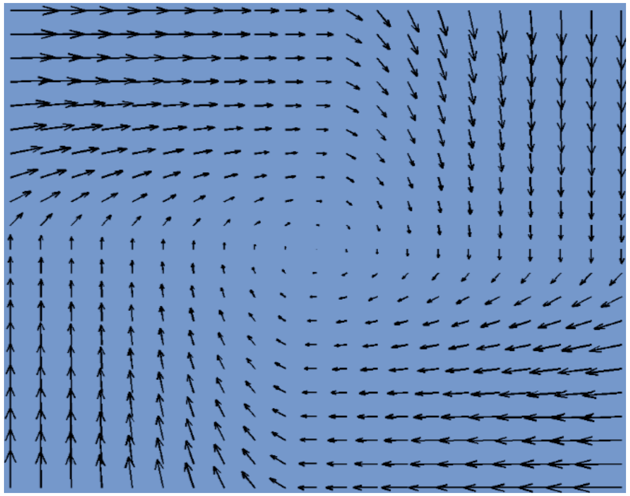
1, -1	**-1,1**
-1, 1	**1,-1**

Draw the vector field for $f(x) - x$.



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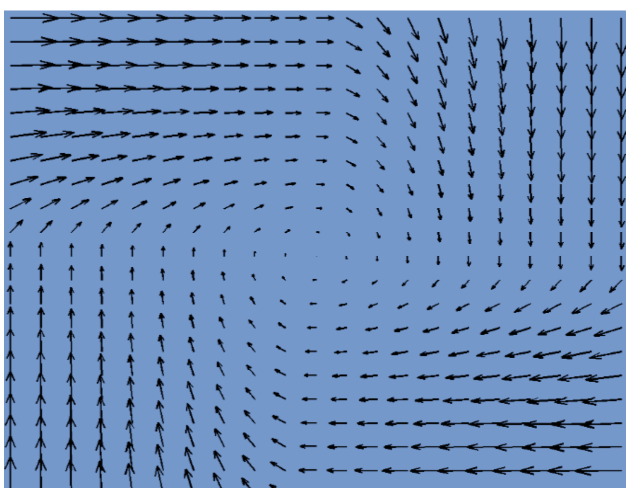

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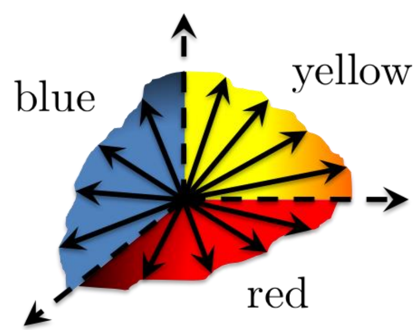

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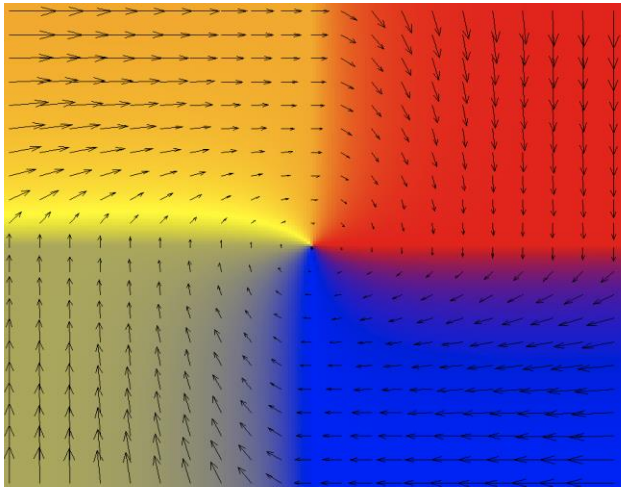

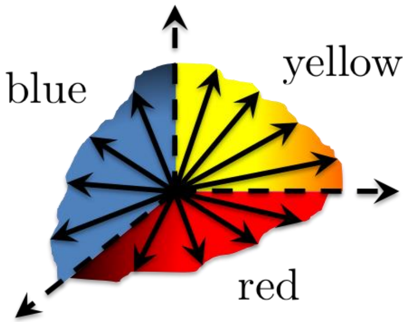

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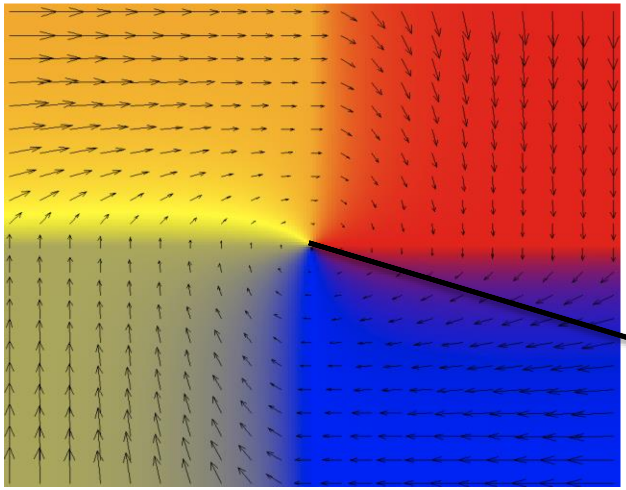

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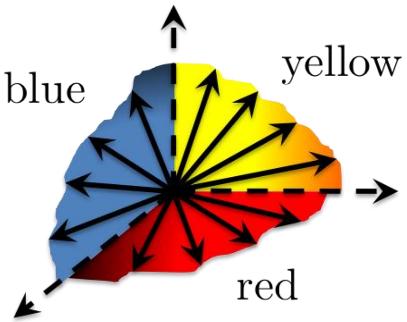
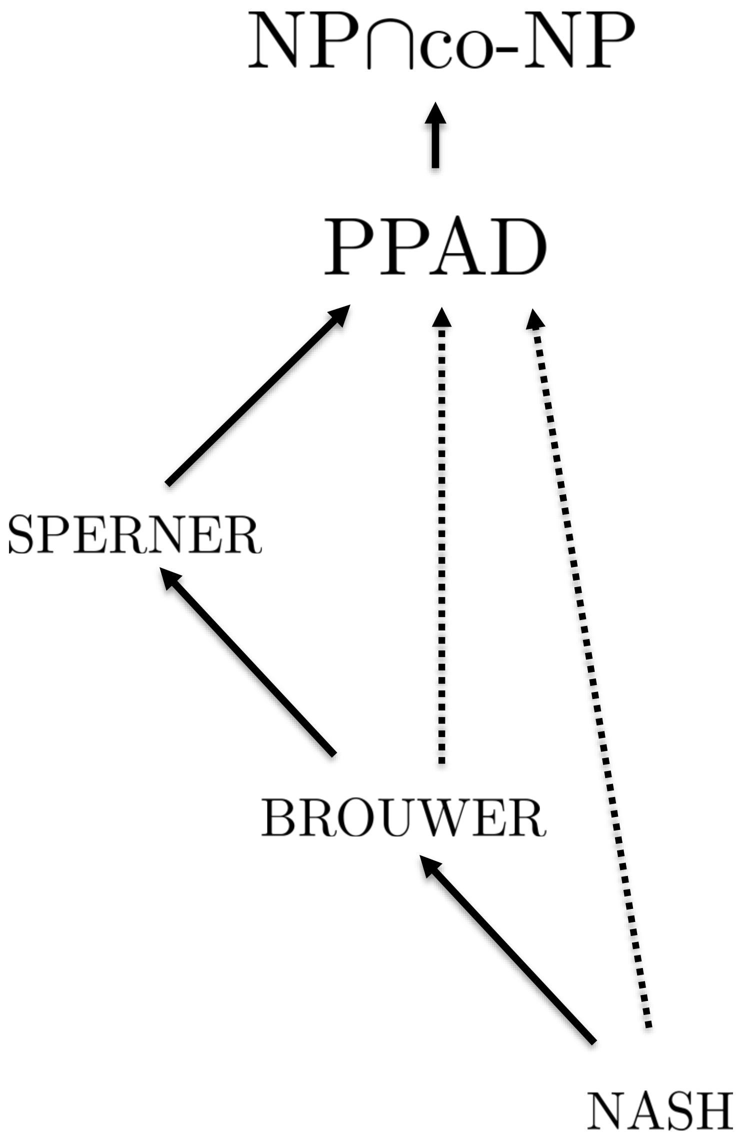


Nash equilibrium $(\frac{1}{2}, \frac{1}{2})$!

Inclusions we showed



Theorem ((NASH is PPAD-complete) Daskalakis, Goldberg, Papadimitriou).
NASH is PPAD-complete.

$$\begin{tikzcd}
\mathrm{PPAD} \arrow[r] & \mathrm{SPERNER} \arrow[r] & \mathrm{BROUWER} \arrow[dl] \\
& \mathrm{NASH} &
\end{tikzcd}$$

Inclusions: The full picture